Brauer's class number relation for the S-ideal class number of an algebraic number field

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Abstract. Let K/k be a Galois extension of algebraic number fields with a Galois group G. Let Γ be the set of all the subgroups of G. Let S be a finite set of prime ideals of k. Denote by $h_S(H)$ the S-class number of K^H. Let ψ_H be the induced character from the trivial character of H. If a $\boldsymbol{Z}$-linear combination $\sum_{H\in\Gamma} n_H\psi_H$ equals 0, we shall show a formula giving the value of $\delta(h_S) = \prod_{H\in\Gamma} h_S(H)^{n_H}$ (Brauer's class number relation) and shall study its applications when G is an abelian p-group for a prime number p.

1. Introduction. The trivial character 1_H of the subgroup of a finite group G induces a character of G, which is called an induced character. We denote this character by ψ_H. More concretely, it is the character afforded with a $\boldsymbol{Q}[G]$-module $\boldsymbol{Q}[G/H] = \boldsymbol{Q}[G]\otimes_{\boldsymbol{Q}[H]}\boldsymbol{Q}$. If a $\boldsymbol{Z}$-linear combination of ψ_H is equal to 0 as a function on G, we call a relation

$$\sum_{H\in\Gamma} n_H\psi_H = 0, \tag{1}$$

a character relation, where Γ is the set consisting of every subgroup of G. We are interested in this relation if it is non-trivial. Let ψ_+ (*resp.* ψ_-) be the partial sum of $n_H\psi_H$ such that $n_H > 0$ (*resp.* $n_H < 0$). We have $\psi_+ = \psi_-$.

We suppose G is the Galois group of a Galois extension K/k of algebraic number fields. Let $h(H)$ denote the class number of the intermediate field corresponding to a subgroup H. Then, associated with the character relation (1), we define $\delta(h)$ by

$$\delta(h) = \prod_{H\in\Gamma} h(H)^{n_H}.$$

The class number relation with respect to (1) is a formula describing the value $\delta(h)$, coming from Artin's L-functions $L(s,\psi_H)$'s. Namely, it is well-known that $L(s,\psi_H)$ coincides with the Dedekind's zeta function $\zeta_{K^H}(s)$ and has a multiplicative property

$$L(s,\psi_H+\psi'_H) = L(\psi_H)L(\psi'_H),$$

c.f. [8, Chapter 0]. Therefore, we obtain a relation of Artin's L-function, and further obtained that of zeta functions. The class number relation yields by taking residue at

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$s = 1$, *c.f.* [5, §1]. This class number relation contains a term concerning regulators of subfields K^H's. This term can be removed. For instance, when $k = \boldsymbol{Q}$, there is a unit ϵ of K such that $\boldsymbol{Z}[G]\epsilon \cong \boldsymbol{Z}[G]/\boldsymbol{Z}s_G$ holds for $s_G = \sum_{\sigma \in G} \sigma$. Thus, the unit group E of K contains a subgroup M which is isomorphic to $\boldsymbol{Z}[G]/\boldsymbol{Z}s_G$. In this case, the following formula of $\delta(h)$ was obtained:

$$(2) \qquad \delta(h) = \prod_{H \in \Gamma} ([E^H : M^H][G : H])^{n_H}$$

c.f. [9, Theorem 4.1]. It was proved in [9] that this formula is also valid for an arbitrary Galois extension K/k in [9].

On the other hand, another generalization was showed in [5]. Let S be a finite set of prime ideals of k. Denote by $S(H)$ the set of every primes of K^H lying above every primes contained in S. The S-ideal class group of K^H is the quotient group of the ideal class group of K^H by a subgroup generated by every prime ideals contained in $S(H)$. Denote by $h_S(H)$ the order of the S-ideal class group. Then, it was shown in [5, Theorem 2.7] that a class number relation holds for the S-class number. However, it contains terms concerning S-regulators. The aim of the present paper is transform this class number relation to the similar formula describing $\delta(h_S)$ as (2) by applying the theory of hermitian $\boldsymbol{Z}[G]$-modules developed in [5]. The formula is given in Theorem 3 in §4 below. In §5, we obtain a special character relation for abelian p-group G, where p is a prime number. This relation is a generalization of the character relation for $G = (\boldsymbol{Z}/p\boldsymbol{Z})^m$ studied in [10]. In §6, we give two examples of class number relations deducing from this character relation.

2. A symmetric $\boldsymbol{Z}[G]$-module. Let $\boldsymbol{Z}[G]$ be the group ring of a finite group G over the ring $\boldsymbol{Z}$ of integers. A finitely generated torsion free $\boldsymbol{Z}[G]$-module is called a $\boldsymbol{Z}[G]$-lattice. The contragredient module of a $\boldsymbol{Z}[G]$-lattice M is a $\boldsymbol{Z}[G]$-lattice $\mathrm{Hom}(M, \boldsymbol{Z})$. We identify M^{**} to M canonically, *c.f.* [2, §10.D]. Let V be the $\boldsymbol{R}[G]$-module obtained by extension of coefficients to the field $\boldsymbol{R}$ of real numbers: $V = M \otimes \boldsymbol{R}$. The $\boldsymbol{R}$-contragredient V^* is the dual space as an $\boldsymbol{R}$-linear space. An $\boldsymbol{R}[G]$-homomorphism $h_V : V \to V^*$ defines a G-invariant bilinear $\boldsymbol{R}$-form on $V \times V$, which is given by

$$(3) \qquad \langle u, v \rangle = h_V(u)(v), \quad u, v \in V.$$

This form is non-degenerate if and only if h_V is an $\boldsymbol{R}$-isomorphism. Conversely, if V has a G-invariant form, an $\boldsymbol{R}[G]$-homomorphism h_V is defined by (3). This notion was generalized to $\boldsymbol{Z}[G]$-modules in [5]. Let M be a finitely generated $\boldsymbol{Z}[G]$-module. We denote by M_{tor} the maximal torsion submodule. We see the quotient module $\bar{M} = M/M_{tor}$ is a $\boldsymbol{Z}[G]$-lattice. So we obtain an $\boldsymbol{R}[G]$-module $V = \bar{M} \otimes \boldsymbol{R}$, which contains $\bar{M}$ as a full sublattice. Since an isomorphism of $M \otimes \boldsymbol{R}$ onto V is induced from the canonical map $\bar{\imath} : M \to \bar{M}$, we identify $M \otimes \boldsymbol{R}$ with V by this isomorphism. If there is a $\boldsymbol{Z}[G]$-homomorphism $h : M \to V^*$, $\bar{i}$ factors h. Take the map $\bar{h} : \bar{M} \to V^*$ so that $h = \bar{h} \circ \bar{i}$ holds. Thus, an $\boldsymbol{R}[G]$-homomorphism $\bar{h}_V : V \to V^*$ is yielded. A bilinear $\boldsymbol{R}$-form is obtained by means of (3) from $\boldsymbol{Z}[G]$-lattice $\bar{M}$. We abuse notation and denote by $h(u,v)$ this G-invariant bilinear $\boldsymbol{R}$-form on V. According to [5, Définition 2.1], the pair (M,h) is called *an $\boldsymbol{R}$-valued hermitian $\boldsymbol{Z}[G]$-module.* However,we say (M,h) is *an $\boldsymbol{R}$-valued symmetric $\boldsymbol{Z}[G]$-module* or *a symmetric $\boldsymbol{Z}[G]$-module* in short, because we study the case that the $\boldsymbol{R}$-form $h(u,v)$ is a symmetric form. We note that this form is non-degenerate if and only if $\bar{h}$ is injective. Let r be the rank of $\bar{M}$. If $\bar{h}$ is injective, the Gram matrix

$$(h(m_i, m_j))_{1\le i,j\le r}$$

is defined for an arbitrary $\boldsymbol{Z}$-basis $\{m_i : 1 \le i \le r\}$ of $\bar{M}$. The discriminant of the symmetric $\boldsymbol{Z}[G]$-module $(\bar{M},\bar{h})$ is defined to be the absolute value of the determinant of the Gram matrix,and is denoted by $\mathrm{disc}(\bar{M},\bar{h})$. In general, the discriminant of a symmetric $\boldsymbol{Z}[G]$-module (M,h) is defined to be

$$\mathrm{disc}(M,h) = \frac{\mathrm{disc}(\bar{M},\bar{h})}{|M_{tor}|}, \tag{4}$$

c.f. [5, Definition 2.3]. A morphism of a symmetric $\boldsymbol{Z}[G]$-module (M_1,h_1) into (M_2,h_2) is a $\boldsymbol{Z}[G]$-homomorphism $\sigma : M_1 \to M_2$ satisfying the relation $h_1(u,v) = h_2(\sigma u, \sigma v)$ for $u, v \in V_1^*$, where $V_1 = M_1 \otimes \boldsymbol{R}$. If σ is an isomorphism, we call it an isometry and say that (M_1,h_1) is isometric to (M_2,h_2). A direct sum and a tensor product of two modules are defined by means of functorial isomorphisms:

$$\begin{aligned} (V_1 \oplus V_2)^* &\cong V_1^* \oplus V_2^* \\ (V_1 \otimes_{\boldsymbol{R}} V_2)^* &\cong \mathrm{Hom}_{\boldsymbol{R}}(V_2, V_1^*) \\ &\cong V_2^* \otimes_{\boldsymbol{R}} V_1^* \cong V_1^* \otimes_{\boldsymbol{R}} V_2^* \end{aligned}$$

c.f. [2, Proposition 10.30]. We identify $(V_1 \oplus V_2)^*$ (*resp.* $(V_1 \otimes_{\boldsymbol{R}} V_2)^*$ with $V_1^* \oplus V_2^*$ (*resp.* $V_1^* \otimes_{\boldsymbol{R}} V_2^*$) by these isomorphisms. h_1 and h_2 induce $\boldsymbol{Z}[G]$-homomorphisms

$$\begin{aligned} h_1 \oplus h_2 &: M_1 \oplus M_2 \to V_1^* \oplus V_2^* \\ h_1 \otimes h_2 &: M_1 \otimes M_2 \to V_1^* \otimes_{\boldsymbol{R}} V_2^*. \end{aligned}$$

These symmetric $\boldsymbol{Z}[G]$-modules is denoted by $(M_1 \oplus M_2, h_1 \oplus h_2)$ and $(M_1 \otimes M_2, h_1 \otimes h_2)$, respectively. Let H be a subgroup of G. $\bar{h}_V$ maps the submodule V^H of H-invariant elements into V^{*H}. We have $(V^H)^* = (V^*)^H$ by [2, Proposition 10.28]. Since $\overline{M^H}$ is a submodule $(\bar{M})^H$ of finite index, we have $\overline{M^H} \otimes \boldsymbol{R} = (\bar{M})^H \otimes \boldsymbol{R} = V^H$. Thus, if we define a homomorphism h^H by

$$\frac{1}{|H|} h : M^H \to V^{*H},$$

the pair (M^H, h^H) is a symmetric $\boldsymbol{Z}[\{1\}]$-module. Denote this symmetric $\boldsymbol{Z}$-module by $(M,h)^H$ in short, *c.f.* [5, Notation 4.8].

The group ring is provided with involution

$$\left(\sum_{\sigma\in G} a_\sigma \sigma\right)^* = \sum_{\sigma\in G} a_\sigma \sigma^{-1}.$$

We have $(xy)^* = y^*x^*$ for product of two elements x and y of the group ring. A non-degenerate G-invariant symmetric bilinear $\boldsymbol{R}$-form $\langle x, y\rangle$ on $\boldsymbol{R}[G]$ is defined from the trivial character 1_G of G:

$$\langle x, y\rangle = 1_G(y^*x) \tag{5}$$

Hereafter, we denote by the symbol V this symmetric $\boldsymbol{R}$-space $\boldsymbol{R}[G]$. V is self-dual, that is $V^* = V$. We denote a sum of every element contained in a subset A of G by s_A or $s(A)$. An idempotent element associated to the subgroup H in the group ring $\boldsymbol{R}[G]$ is defined to be

$$e_H = \frac{1}{|H|}s_H. \tag{6}$$

Denote by $\boldsymbol{Z}[G]e_H$ a $\boldsymbol{Z}[G]$-submodule generated by e_H. Put $V_H = \boldsymbol{R}[G]e_H$. We define a non-degenerate G-invariant symmetric bilinear $\boldsymbol{R}$-form h_H by

$$h_H(u, v) = |H| < u, v >$$

on V_H. We see $h_H(\sigma e_H, \sigma e_H) = 1$ and $h_H(\sigma e_H, \tau e_H) = 0$ if $\sigma e_H \neq \tau e_H$. Thus, we have V_H is self-dual with respect to h_H. The form h_H is considered it is induced from inclusion $\boldsymbol{Z}[G]e_H \to V_H$. The inclusion map gives a symmetric $\boldsymbol{Z}[G]$-module structure. We also denote this structure by h_H:

$$h_H : \boldsymbol{Z}[G]e_H \to V_H = (V_H)^*. \tag{7}$$

Moreover, since $\{\sigma e_H\}$ is a $\boldsymbol{Z}$-basis of $\boldsymbol{Z}[G]e_H$, the symmetric $\boldsymbol{Z}[G]$-module $(\boldsymbol{Z}[G]e_H, h_H)$ is unimodular, *c.f.* [5, Notation 5.14].

The following lemma is a consequence from Corollary 4.14 in [5]. We shall give an elementary proof following to the proof of Proposition 10.31 in [2].

LEMMA 1. *Let (M, h) be an arbitrary non-degenerate symmetric $\boldsymbol{Z}[G]$-module. Then, we have an isometry*

$$(\boldsymbol{Z}[G]e_H \otimes M, h_H \otimes h)^G \to (M, h)^H.$$

Proof. Let $[G/H]$ be the complete set of representatives of right cosets. The set $\{\sigma e_H : \sigma \in [G/H]\}$ is a $\boldsymbol{Z}$-basis of the free $\boldsymbol{Z}$-module $\boldsymbol{Z}[G]e_H$. Thus, each element x is written uniquely as a sum

$$x = \sum_{\sigma\in[G/H]} \sigma e_H \otimes m_{\bar\sigma}, \quad m_{\bar\sigma} \in M,$$

where $\bar\sigma$ denotes the right coset σH. If x is G-invariant, we see

$$gx = \sum_{\sigma} g\sigma e_H \otimes gm_{\bar\sigma} = x$$

for every $g \in G$. Since the coefficient $m_{\bar\sigma}$ of each σ is uniquely determined for x, we have $m_{\bar g} = gm_{\bar 1}$. In particular, if we set $g \in H$, we have $m_{\bar 1} = gm_{\bar 1}$. Thus, by sending $x \in (\boldsymbol{Z}[G]e_H \otimes M)^G$ to $m_{\bar 1} \in M^H$, an injective mapping is defined. It is easy to verify this mapping is a surjective homomorphism. Therefore, $(\boldsymbol{Z}[G]e_H \otimes M)^G \cong M^H$ as $\boldsymbol{Z}$-modules. We shall show this isomorphism is an isometry. Let x and y be two elements of $(\boldsymbol{Z}[G]e_H \otimes M)^G$:

$$x = \sum_{\sigma \in G/H} \sigma e_H \otimes \sigma m,$$
$$y = \sum_{\sigma \in G/H} \sigma e_H \otimes \sigma n$$

for $m, n \in M^H$. Denote by $\bar{m}$ and $\bar{n}$ the images into $\bar{M} \otimes \boldsymbol{R}$. We have

$$\begin{aligned} &\tfrac{1}{|G|} \cdot h_H \otimes h(x)(y) \\ &= \tfrac{1}{|G|} \sum_{\sigma,\tau \in [G/H]} h_H(\sigma e_H)(\tau e_H) \cdot h(\sigma \bar{m})(\tau \bar{n}). \\ &= \tfrac{1}{|G|} \sum_{\sigma \in [G/H]} h(\sigma \bar{m})(\sigma \bar{n}) \\ &= \tfrac{1}{|H|} h(\bar{m})(\bar{n}). \end{aligned}$$

This shows the isomorphism is an isometry. □

The subset consisting of every H such that $n_H > 0$ (*resp.* $n_H < 0$) is denoted by Γ_+ (*resp.* Γ_-). Associated to these subsets, we define $\boldsymbol{Z}[G]$-modules $M_\pm$ to be $M_\pm = \bigoplus_{H \in \Gamma_\pm} (\boldsymbol{Z}[G]e_H)^{|n_H|}$. The non-degenerate symmetric $\boldsymbol{Z}[G]$-module structures are defined on $M_\pm$ by

$$(8) \quad (M_\pm, h_\pm) = \bigoplus_{H \in \Gamma_\pm} (\boldsymbol{Z}[G]e_H, h_H)^{n_H}$$

from (7). Since the character relation (1) asserts there is an $\boldsymbol{Q}[G]$-isomorphism

$$M_+ \otimes \boldsymbol{Q} \cong M_- \otimes \boldsymbol{Q},$$

there is an injective $\boldsymbol{Z}[G]$-homomorphism of M_- into $M_+ \otimes \boldsymbol{Q}$. Put $V_\pm = M_\pm \otimes \boldsymbol{R}$. Let j be an $\boldsymbol{R}[G]$-isomorphism of V_- into V_+ obtained from this $\boldsymbol{Z}[G]$-homomorphism. Let j^* be the adjoint of j with respect to $\boldsymbol{R}$-forms on $V_\pm$. Namely, j^* is defined by

$$(9) \qquad h_+(j(u), v) = h_-(u, j^*(v)).$$

j^* is an $\boldsymbol{R}[G]$-isomorphism of V_+ onto V_-.

By [5, §5.4], the fundamental invariant $\delta(M_+, M_-; M)$ is defined for an arbitrary non-degenerate symmetric $\boldsymbol{Z}[G]$-module (M, h). We write it as $\delta(M, h)$ or $\delta(M)$ in short. The following discriminant relation holds from [5, Theorem 6.1]:

$$(10) \quad \delta(M, h) = \frac{\operatorname{disc}((M_+ \otimes M, h_+ \otimes h)^G)}{\operatorname{disc}((M_- \otimes M, h_- \otimes h)^G)}.$$

If we define a function f on Γ by $f(H) = \operatorname{disc}((M, h)^H)$, we have by virtue of Lemma 1 a formula

$$\delta(M) = \prod_{H \in \Gamma} f(H)^{n_H}.$$

We generalize this notion to an arbitrary function f taking values in non-zero real numbers. We define a functional δ on such f's to be

$$\delta(f) = \prod_{H \in \Gamma} f(H)^{n_H}.$$

This functional is multiplicative. When f is a constant function, we see $\delta(f) = f(1)^{\sum n_H}$. However, this value $\delta(f)$ equals to 1, because we have

$$\begin{aligned} \langle 1_G, \sum_{H \in \Gamma} n_H \psi_H \rangle_G &= \sum_{H \in \Gamma} n_H \langle 1_G, \psi_H \rangle_G \\ &= \sum_{H} n_H \langle 1_H, 1_H \rangle_H \end{aligned}$$

from the Frobenius reciprocity law, *c.f.* [1, (4)].

REMARK 1. If M is a $\boldsymbol{Z}[G]$-module of finite order, it has a trivial symmetric $\boldsymbol{Z}[G]$-module structure, because of $M \otimes \boldsymbol{R} = 0$. We denote this structure by $(M, *)$. Note

$$\delta(M,*) = \prod_{H \in \Gamma} \frac{1}{|M^H|^{n_H}}$$

from the definition (4).

3. The group of S-units. We assume G is the Galois group of a finite Galois extension K/k of algebraic number fields. Let S be a finite set of places of k containing all the archimedian places. Denote by S_0 the subset of every non-archimedian places. Suppose $S = \{v_1, \cdots, v_s\}$. We choose a prolongation onto K of each v_i and fix it once for all. Denote by w_i the selected place. Let G_i be the decomposition group of w_i. Every place of K^H lying above v_i is obtained from the decomposition into two sided cosets:

$$G = \dot{\cup}_{j=1}^{s_i} H\sigma_{ij}G_i.$$

Let u_i be restriction of w_i onto K^H. There are s_i places $\sigma_{ij}u_i, j = 1, \cdots, s_i$ over v_i. Denote by $S_0(K^H)$ the union of all such places for every $v_i \in S_0$. Let $\mathcal{P}_k$ be the set of the all places of k. Denote by $|\cdot|_v$ be the normalized multiplicative valuation for $v \in \mathcal{P}_k$ so that the product formula holds. Namely,

$$\prod_{v \in \mathcal{P}_k} |x|_v = 1$$

holds for every $x \in k^\times$. Further, we associate a multiplicative valuation $||\cdot||_{w_i}$ to each w_i so that the value $||x||_{w_i}$ for every $x \in k$ argrees to the value $|x|_{v_i}$. Denote by $h_S(H)$ the order of the S-ideal class group of K^H. $h_S(H)$ is a function on Γ. An element $x \in K$ is called an S-unit if an arbitrary prime divisor of the principal ideal (x) belongs to the set of valuation ideals of places contained in $S_0(K)$. The subgroup of $K^\times$ generated by every S-unit of K is called the group of S-units of K and is denoted by E_S. We shall give two non-degenerate symmetric $\boldsymbol{Z}[G]$-module structures on E_S. We abbreviate the idempotent e_{G_i} defined by (6) to e_i. Put

$$\begin{aligned} L_S &= \bigoplus_{i=1}^{s} Z[G]e_i, \\ V_S &= \bigoplus_{i=1}^{s} Ve_i. \end{aligned}$$

A non-degenerate symmetric $\boldsymbol{R}$-form on V_S is defined by

$$\langle \sum_{i=1}^{s} u_i, \sum_{i=1}^{s} v_i \rangle = \sum_{i=1}^{s} h_{G_i}(u_i, v_i). \tag{11}$$

The inclusion map of L_S into V_S is given by

$$h_S = \oplus_{i=1}^{s} h_{G_i} : L_S \to V_S = V_S^*,$$

which is a non-degenerate symmetric G-invariant $\boldsymbol{Z}[G]$-module structure on L_S. Let $[G/G_i]$ be a complete set of representatives of G/G_i. Put $\alpha_i = s([G/G_i])$. L_S^G is a free $\boldsymbol{Z}$-module on a basis $\{\alpha_i e_i : 1 \leq i \leq s\}$. Put $\eta = (\alpha_1 e_1, \cdots, \alpha_s e_s) \in L_S^G$. V_S^G contains a one-dimensional subspace generated by η. Since $L_S \cap V_\eta = \boldsymbol{Z}\eta$, there is an injective $\boldsymbol{Z}[G]$-homomorphism $L_S/\boldsymbol{Z}\eta \to$

V_S/V_η. Moreover, V_η has an orthogonal complement $V_{S,1}$ in V_S:

$$V_{S,1} = \{u \in V_S : \langle \eta, u\rangle = 0\}$$

with respect to (11). We observe

$$\langle \sum_{i=1}^{s} \sum_{\sigma_i \in [G/G_i]} a_{\sigma_i}\sigma_i e_i, \eta\rangle = \sum_{i=1}^{s} \sum_{\sigma_i \in [G/G_i]} a_{\sigma_i}.$$

Thus, if we define $|u| = \langle u, \eta\rangle$, we see $u \in V_{S,1}$ is equivalent to $|u| = 0$. We consider $V_{S,1}$ as a symmetric space by restricting the $\boldsymbol{R}$-form on V_S. Since this symmetric form is G-invariant, we have $V_{S,1}^* = V_{S,1}$ as $\boldsymbol{R}[G]$-modules with respect to the symmetric form. Let $h_{S,1}$ be the composite map of the canonical map $V_S/V_\eta \to V_{S,1}$ induced from the projection onto $V_{S,1}$ and the homomorphism of $L_S/\boldsymbol{Z}\eta$ into V_S/V_η. Put $L_{S,1} = L_S/\boldsymbol{Z}\eta$. The pair $(L_{S,1}, h_{S,1})$ is a non-degenerate symmetric $\boldsymbol{Z}[G]$-module.

We apply the generalized Dirichlet-Herbrand theorem on S-units, *c.f.* [4, Theorem I.3.7]. There is a $\boldsymbol{Q}[G]$-isomorphism

$$E_S \otimes \boldsymbol{Q} \to L_{S,1} \otimes \boldsymbol{Q}. \tag{12}$$

Thus, $E_S \otimes \boldsymbol{R} \cong V_{S,1}$. Since E_S is mapped into $E_S \otimes \boldsymbol{R}$ by $x \to x \otimes 1$, there is a $\boldsymbol{Z}[G]$-homomorphism h of E_S into $V_{S,1}$. This makes E_S a non-degenerate symmetric $\boldsymbol{Z}[G]$-module.

E_S is provided with another non-degenerate symmetric $\boldsymbol{Z}[G]$-module structure. Let $l : E_S \to V_S$ be a $\boldsymbol{Z}[G]$-homomorphism defined to be

$$l(u) = \left(\sum_{\sigma \in G} \log ||\sigma^{-1}u||_{w_i} \sigma e_i\right)_{1 \le i \le s}.$$

We see

$$| \, l(u) \, | = \log \left(\prod_{i=1}^{s} \prod_{\sigma_i \in [G/G_i]} ||\sigma_i^{-1}u||_{w_i}^{|G_i|} \right).$$

The product formula of the multiplicative valuations normalized to the algebraic number field K asserts this value is equal to 0. Hence, l takes values in $V_{S,1}$. Since S contains every archimedian place, $\operatorname{Ker} l = E_{tor}$. Thus, $\boldsymbol{Z}[G]$-module (E_S, l) is a non-degenerate symmetric. We compute $\langle l(u), l(v)\rangle$ and obtain

$$\langle l(u), l(v)\rangle = \sum_{i=1}^{s} \sum_{\sigma_i \in [G/G_i]} \log ||\sigma_i^{-1}u||_{w_i} \log ||\sigma_i^{-1}v||_{w_i} |G_i|. \tag{13}$$

This shows $\langle l(u), l(v)\rangle$ coincides with the form $\rho_S(u, v)$ defined in [5, (8.1)]. We restate here the following formula obtained in [5, Theorem 2.7]:

THEOREM 2 (Kani). *Let w be the function on Γ defined to be $w(H) = |E_{S,tor}^H|$. Then, we have*

$$\delta(h_S)^2 = \frac{\delta(\boldsymbol{Z})\delta(w)}{\delta(E_S, l)\delta(L_S)}.$$

REMARK 2. If H is cyclic, we have $\delta(\boldsymbol{Z}[G]e_H) = 1$ by [5, Example 2.13. a)]. Therefore, $\delta(\boldsymbol{Z}[G]e_i) = 1$ if v_i is archimedian. We see $\delta(L_S) = \delta(L_{S_0})$.

REMARK 3. We define a function n_G on Γ to be $n_G(H) = |G : H|$. We have

$$\delta(\boldsymbol{Z}) = \prod_{H \in \Gamma} |H|^{-n_H} = \delta(n_G),$$

c.f. [5, (2.7)].

REMARK 4. Let w_2 be the 2-part of w. We have $\delta(w) = \delta(w_2)$ from [1, §2.5].

4. Brauer's class number relations. $V_{S,1}^* = V_{S,1}$ contains a $\boldsymbol{Z}[G]$-lattice isomorphic to $L_{S,1} = L_S/\boldsymbol{Z}\eta$. The inverse image M' by $h : E_S \to V_{S,1}^*$ of the lattice is a submodule containing $E_{S,tor}$. Since $\operatorname{Ker} h = E_{S,tor}$, $M'^{|E_{S,tor}|}$ is torsion free and is isomorphic to $L_{S,1}$. Hence, E_S contains a $\boldsymbol{Z}[G]$-submodule isomorphic to $L_{S,1}$. Let M_S be an arbitrary such $\boldsymbol{Z}[G]$-submodule. By restricting the two symmetric $\boldsymbol{Z}[G]$-module structures of E_S, M_S is also provided with two structures. We denote them by (M_S, h) and (M_S, l), respectively. We shall prove the following class number relation holds:

THEOREM 3. *We define a function i_{E_S,M_S} on Γ to be $i_{E_S,M_S}(H) = [E_S^H : M_S^H]$. Then, we have*

$$\delta(h_S) = \frac{\delta(i_{E_S,M_S})\delta(n_G)}{\delta(L_{S_0})}.$$

This theorem is a generalization to S-class numbers of Brauer's class number relation proved in [9, Theorem 4.1]. The key of the proof is the following lemma:

LEMMA 4. $\delta(M_S, h) = \delta(M_S, l)$.

Proof. Let ι (*resp.* $\iota_\pm$) be identity map (*resp.* identity maps) on $V_{S,1}^*$ (*resp.* $V_\pm^*$). Since the adjoint map j^* in (9) is an isomorphism, $j^* \otimes \iota$ is an isomorphism of $V_+^* \otimes_{\boldsymbol{R}} V_{S,1}^*$ onto $V_-^* \otimes_{\boldsymbol{R}} V_{S,1}^*$. Denote by $(j^* \otimes \iota)(G)$ restriction of $(j^* \otimes \iota)$ on the G-invariant submodules. Let α be an automorphism on $V_{S,1}^*$ which is induced from an isomorphism $l \circ h^{-1} : h(M_S) \to l(M_S)$ of sublattices. We abbreviate $\iota_\pm \otimes \alpha$ to $\alpha_\pm$ in short and denote by $\alpha_\pm(G)$ restriction onto $(V_\pm^* \otimes_{\boldsymbol{R}} V_{S,1}^*)^G$. We have

$$\alpha_+(G) = (j^* \otimes \iota)(G)^{-1} \circ \alpha_-(G) \circ (j^* \otimes \iota)(G).$$

Thus, $\det \alpha_+(G) = \det \alpha_-(G)$. Concerning two symmetric $\boldsymbol{Z}[G]$-module structures $(h_\pm \otimes h)^G$ and $(h_\pm \otimes l)^G$ on $(M_\pm \otimes M_S)^G$, we have the following commutative diagram:

$$\begin{array}{ccc} (M_\pm \otimes M_S)^G & \xrightarrow[(h_\pm \otimes h)^G]{} & (V_\pm^* \otimes V_{S,1}^*)^G \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \alpha_\pm(G)} \\ (M_\pm \otimes M_S)^G & \xrightarrow[(h_\pm \otimes l)^G]{} & (V_\pm^* \otimes V_{S,1}^*)^G. \end{array}$$

Thus, a relation between the Gram matrices

$$\begin{aligned} &\operatorname{disc}((M_\pm \otimes M_S, h_\pm \otimes l)^G) = \\ &\quad (\det \alpha_\pm(G))^2 \operatorname{disc}((M_\pm \otimes M_S, h_\pm \otimes h)^G). \end{aligned}$$

is obtained. Hence, it follows $\delta(M_S, h) = \delta(M_S, l)$ from (10). □

Proof of Theorem 3. The quotient module E_S/M_S is of finite order. It is a trivial non-degenerate symmetric $\boldsymbol{Z}[G]$-module $(E_S/M_S, *)$. Thus, we have an exact sequence in the category of symmetric $\boldsymbol{Z}[G]$-modules:

$$(14)\quad 1 \to (M_S, h) \to (E_S, h) \to (E_S/M_S, *) \to 1.$$

For each $H \in \Gamma$, the following sequence is exact:

$$1 \to \boldsymbol{Z}[G]e_H \otimes M_S \to \boldsymbol{Z}[G]e_H \otimes E_S \to \boldsymbol{Z}[G]e_H \otimes E_S/M_S \to 1.$$

We have a cohomology long exact sequence

$$1 \to (\boldsymbol{Z}[G]e_H \otimes M_S)^G \to (\boldsymbol{Z}[G]e_H \otimes E_S)^G \to (\boldsymbol{Z}[G]e_H \otimes E_S/M_S)^G \xrightarrow{\delta_H} H^1(G, \boldsymbol{Z}[G]e_H \otimes M_S)$$

from this sequence. We can apply "exact sequence formula", *c.f.* [5, Theorem 6.21]. We have

$$(15)\quad \delta(E_S, h)\delta(E_{S,tor}) = \delta(M_S, h)\delta(E_S/M_S)^2\psi^2,$$

where

$$\psi = \prod_{H\in\Gamma} |\mathrm{Im}\ \delta_H|^{n_H}.$$

We notice that the first three terms in a cohomology long exact sequence

$$1 \to M_S^H \to E_S^H \to (E_S/M_S)^H \xrightarrow{f_H} H^1(H, M_S)$$

are equal to those in the above cohomology exact sequence by virtue of Lemma 1. Hence, $|\mathrm{Im}\ \delta_H| = |\mathrm{Im}\ f_H|$ and

$$\frac{|\mathrm{Im}\ \delta_H|}{|(E_S/M_S)^H|} = \frac{|\mathrm{Im}\ f_H|}{|(E_S/M_S)^H|}.$$

Therefore, we obtain

$$(16)\qquad \frac{1}{[E_S^H : M_S^H]} = \frac{|\mathrm{Im}\ \delta_H|}{|(E_S/M_S)^H|}.$$

and an auxiliary formula

$$(17)\qquad \delta(E_S/M_S)\psi = \delta(i_{E_S,M_S})^{-1}$$

Moreover, since $L_S/\boldsymbol{Z}\eta \cong M_S$, we also have

$$(18)\qquad \delta(L_S) = \delta(n_G)\delta(M_S, h)$$

from "exact sequence formula". Combining (15), (17) and (18), we have

$$\frac{\delta(E_S, h)}{\delta(w)} = \frac{\delta(L_S)}{\delta(n_G)\delta(i_{E_S,M_S})^2}$$

because of $\delta(E_{S,tor}) = \delta(w)^{-1}$. Moreover, in account of Lemma 4, we can substitute $\delta(E_S, h)$ for $\delta(E_S, l)$ in the formula of Theorem 2. In consequence, we have a formula

$$\delta(h_S)^2 = \frac{\delta(n_G)^2\delta(i_{E_S,M_S})^2}{\delta(L_S)^2}.$$

This proves the theorem. □

We shall give two applications of Theorem 3.

LEMMA 5. *$\delta(h_S)$ is a unit in the ring $\boldsymbol{Z}_p$ of p-adic integers for every prime number p not dividing $|G|$.*

Proof. Let p be a prime not dividing $|G|$. We see $\delta(n_G) \in \boldsymbol{Z}_p^\times$. By the formula of $\delta(\boldsymbol{Z}[G/H])$ in [5, Example 2.1, b)], we have $\delta(\boldsymbol{Z}[G]e_i) \in \boldsymbol{Z}_p^\times$ for $i = 1, \cdots, s$. Let f_0 be a function on Γ defined by $f_0(H) =$

$|(E_S/M_S)^H|$. Since $p \nmid |H^1(H, M_S)|$, we have from (16) that $\delta(i_{E_S,M_S})$ is a p-adic integer if and only if $\delta(f_0)$ is also. Let Y be the p-primary submodule of E_S/M_S. Let p^m be the exponent of Y. Put $Y_n = Y^{p^n}$ for $n = 0, \cdots, m$. Y_{n-1}/Y_n is an $\boldsymbol{F}_p[G]$-module. Let χ_n be the character of G afforded with an $\boldsymbol{F}_p[G]$-module Y_{n-1}/Y_n. Let $\zeta^{(1)}, \cdots, \zeta^{(r)}$ be a basic set of irreducible $\boldsymbol{Q}_p$-characters of G, where $\boldsymbol{Q}_p$ denotes the field of p-adic numbers. Since $p \nmid |G|$, an $\boldsymbol{F}_p$-irreducible character is obtained from each $\zeta^{(i)}$ by reduction with respect to mod p. Denote by $\bar{\zeta}^{(i)}$ the $\boldsymbol{F}_p$-irreducible character. χ_n is a linear combination of $\bar{\zeta}^{(i)}$'s with non-negative integral coefficients:

$$\chi_n = \sum_{i=1}^{r} c_i \bar{\zeta}^{(i)}.$$

The dimension of $(Y_{n-1}/Y_n)^H$ over $\boldsymbol{F}_p$ is given by the value of

$$\sum_{i=1}^{r} c_i \langle \zeta^{(i)}, \psi_H \rangle_G \dim_{\boldsymbol{F}_p} U_i,$$

where U_i are simple $\boldsymbol{F}_p[G]$-modules affording the characters $\zeta^{(i)}$'s. We see

$$\sum_{H \in \Gamma} n_H \dim_{\boldsymbol{F}_p} (Y_{n-1}/Y_n)^H = 0$$

from (1). Thus, if we define a function f_n on Γ by $f_n(H) = \left|(Y_{n-1}/Y_n)^H\right|$, we have $\delta(f_n) = 1$. Since $f_0(H) = \prod_{n=1}^{m} f_n(H)$, we see $p \nmid \delta(i_{E_S,M_S})$. □

Corollary 6. *Let $h_S^{(p)}(H)$ be the highest power of $h_S(H)$ with respect to a prime p. If $p \nmid |G|$, we have $\delta(h_S^{(p)}) = 1$.*

We assume K is a CM-field and k is a totally real subfield. The Galois group G contains the complex conjugation map τ. By Lemma 8 in the next section, the character relation (1) holds if and only if

$$\sum_{H \in \Gamma} n_H \tilde{e}_H = 0$$

holds. Denote by H^+ a subgroup generated by H and τ. Put $e^+ = \frac{1}{2}(1+\tau)$. Since

$$e_{H^+} = e_H e^+,$$

we have $\sum_{H \in \Gamma} n_H \tilde{e}_{H^+} = 0$ from the above idempotent relation. Thus, by Lemma 8, a character relation $0 = \sum_{H \in \Gamma} n_H \psi_{H^+}$ is yielded. Hence,

$$(19) \qquad 0 = \sum_{H \in \Gamma} n_H (\psi_H - \psi_{H^+}).$$

Let Γ_1 be a subset of Γ consisting of H such that $H \neq H^+$. We define functions $f^{\pm}$ from an arbitrary function f on Γ to be

$$f^-(H) = \frac{f(H)}{f(H^+)} \quad \text{and} \quad f^+(H) = f(H^+).$$

Then, the functional δ' defined from (19) satisfies $\delta'(f^+) = 1$, $\delta'(f) = \delta'(f^-)$ and

$$\delta(f^-) = \delta'(f) = \prod_{H \in \Gamma_1} f^-(H)^{n_H}.$$

Suppose S is the set of all the archimedian places. We put $E_S^+ = E_S^{<\tau>}$. Since $E_S^{2w(1)} \subset E_S^+$, we can choose M_S from a subgroup of E_S^+. We observe an index relation

$$i_{E_S,M_S}(H) = [E_S^H : \mu_K^H E_S^{+H}][\mu_K^H E_S^{+H} : M_S]$$

holds, where μ_K is the subgroup of E_S consisting of every root of unity. Let Q be a function on Γ whose value is equal to the unit index of K^H if $\tau \notin H$ and which takes 1 when $H = H^+$. We have

$$i_{E_S,M_S}(H) = \frac{Q(H)w(H)i_{E_S,M_S}(H^+)}{2}.$$

Therefore, we obtain

COROLLARY 7. *Let K be a CM-field which is a Galois extension on a totally real subfield k. If S is the set of all the archimedian places, then we have*

$$\delta(h_S^-) = \delta(Q^-)\delta(w^-)\delta(n_G^-)$$

with respect to the character relation (1).

REMARK 5. Each of $\delta(Q^-)$, $\delta(w^-)$ and $\delta(n_G^-)$ takes a value of an integral power of 2.

5. Character relations. The induced character ψ_H is defined to be

$$\psi_H(\sigma) = \frac{1}{|H|}\sum_{g\in G} \mathrm{i}_H(g^{-1}\sigma g)$$

where i_H is a function on G taking value 1 for every element of H and taking 0 for elements in $G\backslash H$, *c.f.* [2, (10.3)]. Let χ be a linear combination of ψ_H with integral coefficients n_H. We have

$$\begin{aligned}\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1} &= \sum_{\sigma}\left(\sum_{H\in\Gamma} n_H\psi_H(\sigma)\right)\sigma^{-1}\\ &= \sum_H \frac{n_H}{|H|}\sum_\sigma\sum_{g\in G}\mathrm{i}_H(g^{-1}\sigma g)\sigma^{-1}\\ &= \sum_H \frac{n_H}{|H|}\sum_g\sum_{\sigma\in gHg^{-1}}\sigma^{-1}\\ &= \sum_H n_H\sum_g ge_Hg^{-1}\end{aligned}$$

Put $\tilde{e}_H = \frac{1}{|H|}\sum_{g\in G} e_{gHg^{-1}}$. We have

$$\sum_{\sigma\in G}\chi(\sigma)\sigma^{-1} = \sum_{H\in\Gamma} n_H\tilde{e}_H.$$

Let $\{\zeta^{(1)},\cdots,\zeta^{(r)}\}$ be the basic set of irreducible $\boldsymbol{C}$-characters of G. Put $z = \sum_{H\in\Gamma} n_H\tilde{e}_H$. We have

$$\begin{aligned}\langle\chi,\zeta^{(i)}\rangle_G &= \frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)\zeta^{(i)}(\sigma^{-1})\\ &= \frac{1}{|G|}\zeta^{(i)}\big(\sum_\sigma\chi(\sigma)\sigma^{-1}\big)\\ &= \frac{1}{|G|}\zeta^{(i)}(z).\end{aligned}$$

By [2, Proposition 9.23], we have every class function on $\boldsymbol{C}[G]$ takes value 0 at z if $\langle\chi,\zeta^{(i)}\rangle_G = 0$ for $i = 1,\cdots,r$. Furthermore, this condition implies $z = 0$, because z is an element of the center of $\boldsymbol{C}[G]$. Conversely, if $z = 0$, we also have $\chi = 0$. Thus, we have

LEMMA 8. *The character relation* (1) *holds if and only if $\sum_{H\in\Gamma} n_H\tilde{e}_H = 0$.*

REMARK 6 (norm relations). Let $U(G)$ be a subset of $\boldsymbol{Z}^{|\Gamma|}$ consisting of $\alpha = (\alpha_H)$ such that $\alpha_H s_H = 0$. This subset is a submodule and is called the module of norm relations in [7]. Let Δ_H be the subset of Γ consisting of every cyclic subgroup of G. Denote by $\Delta_{H,U}$ for each cyclic subgroup U the subset $\{N \in \Delta_H : N \geq U\}$. In [7, Satz 1], an element γ^H of $U(G)$ was defined by

$$\gamma_U^H = \begin{cases} 0 & \text{if } U \notin \Delta_G\backslash\{H\},\\ 1 & \text{if } U = H,\\ -\sum_{N\in\Delta_{H,U}}\mu(|N:U|) & \text{if } U \in \Delta_G\backslash\{H\},\end{cases}$$

and it was proved the set $\{\gamma^H : H \notin \Delta_G\}$ is a $\boldsymbol{Z}$-basis of $U(G)$.

REMARK 7 (the formula of Kani-Rosen). Let $R(G)$ be a $\boldsymbol{Q}$-linear subspace of $\boldsymbol{Q}^{|\Gamma|}$ consisting of $\beta = (\beta_H)$ such that $\sum_{H\in\Gamma} \beta_H \psi_H = 0$. Let $\Delta/\sim$ be the set of conjugacy classes of cyclic subgroups. Then, the dimension of $R(G)$ is given by

$$\dim R(G) = |\Gamma| - |\Delta/\sim|$$

c.f. [6, the formula (6)].

Hereafter, we restrict our concern onto abelian p-groups for a fixed prime p. Suppose

$$G \cong \boldsymbol{Z}/p^{m_1}\boldsymbol{Z} \times \cdots \times \boldsymbol{Z}/p^{m_n}\boldsymbol{Z}, \tag{20}$$

for integers $m_1 \geq \cdots \geq m_n \geq 1$. Let $\hat{G}$ be the group of all the characters, that is $\hat{G} = \mathrm{Hom}(G, \boldsymbol{C}^\times)$. Denote by $H^\perp$ (*resp.* $X^\perp$) for a subgroup H (*resp.* X) of G (*resp.* $\hat{G}$) the annihilator of H (*resp.* X). We have $(X^\perp)^\perp = X$ and $H_1^\perp \cap H_2^\perp = (H_1H_2)^\perp$ for subgroups H_1 and H_2 of G. Put $G^* = (\hat{G}^p)^\perp$. We denote by H^* the subgroup HG^* for H. Put $Z = \hat{G}\backslash\hat{G}^p$. Let Γ_0 be the subset of Γ consisting of $\mathrm{Ker}\,\chi$ for every $\chi \in Z$. We have

$$\hat{G} = \bigcup_{H\in\Gamma_0} H^\perp, \quad H^\perp \cap G^{*\perp} = H^{*\perp}.$$

Thus, if $H = \langle\zeta\rangle^\perp$ for $\zeta \in Z$, we have $H^{*\perp} = \langle\zeta^p\rangle$.

LEMMA 9. *If $H_1, H_2 \in \Gamma_0$ are distinct, we have $H_1^\perp \cap H_2^\perp \leq H_1^{*\perp} \cap H_2^{*\perp}$.*

Proof. There are $\zeta_i \in Z$ such that $\langle\zeta_i\rangle = H_i^\perp$. If $\zeta_1 \notin \langle\zeta_2\rangle$, we see $\langle\zeta_1\rangle \cap \langle\zeta_2\rangle \leq \langle\zeta_2^p\rangle$. Thus, we have $H_1^\perp \cap H_2^\perp \leq H_2^{*\perp}$. Similarly, we have $H_1^\perp \cap H_2^\perp \leq H_1^{*\perp}$. □

THEOREM 10. *We have a character relation*

$$\psi_1 - \psi_{G^*} = \sum_{H\in\Gamma_0} (\psi_H - \psi_{H^*}).$$

Proof. By Lemma 9, we have

$$(H_1^\perp\backslash H_1^{*\perp}) \cap (H_2^\perp\backslash H_2^{*\perp}) = \emptyset$$

for every pair $(H_1, H_2) \in \Gamma_0 \times \Gamma_0$ such that $H_1 \neq H_2$. Since $H^{*\perp} = G^{*\perp} \cap H^\perp$, Z is a disjoint union of $H^\perp\backslash H^{*\perp}$:

$$Z = \hat{G}\backslash G^{*\perp} = \bigcup_{H\in\Gamma_0} H^\perp\backslash H^{*\perp}. \tag{21}$$

Every induced character ψ_H is a linear combination of $\zeta \in \hat{G}$ with non-negative integral coefficients m_ζ. The value of m_ζ is computed by the Frobenius reciprocity law:

$$m_\zeta = \langle\psi_H, \zeta\rangle_G = \langle 1_H, \zeta\!\downarrow_H\rangle_H,$$

where $\zeta\downarrow_H$ denotes restriction onto H. Clearly, $m_\zeta = 1$ if $\zeta \in H^\perp$, and $m_\zeta = 0$ if $\zeta\downarrow_H$ is not trivial. Therefore, $\psi_H = \sum_{\zeta\in H^\perp} \zeta$. By (21), we have

$$\begin{aligned}
\psi_1 - \psi_{G^*} &= \sum_{\zeta\in\hat{G}} \zeta - \sum_{\zeta^*\in G^{*\perp}} \zeta^* \\
&= \sum_{H\in\Gamma_0} \left(\sum_{\zeta\in H^\perp} \zeta - \sum_{\zeta^*\in H^{*\perp}} \zeta^* \right) \\
&= \sum_{H\in\Gamma_0} (\psi_H - \psi_{H^*}).
\end{aligned}$$

□

6. Examples.

EXAMPLE 1. Suppose $n \geq 2$ and $m_1 = \cdots = m_n = m \geq 2$ in (20). We have $G^* = G^{p^{m-1}}$ and $\Gamma_0 = \{H : G/H \cong \boldsymbol{Z}/p^m\boldsymbol{Z}\}$. Put $K^* = K^{G^*}$. If $H \in \Gamma_0$ is the kernel of a character χ, we have $H^{*\perp} = \langle \chi^p \rangle$. Thus, we see $H_1^* = H_2^*$ is equivalent to $\langle \chi_1^p \rangle = \langle \chi_2^p \rangle$. If we choose $H \in \Gamma_0$ arbitrarily, the number of subgroups $H' \in \Gamma_0$ such that $H^* = H'^*$ is equal to p^{n-1}. By Theorem 10, we obtain a character relation

$$0 = (\psi_1 - \psi_{G^*}) - \sum_{H \in \Gamma_0} (\psi_H - \psi_{H^*}).$$

Put $\Gamma_0^* = \{H^* : H \in \Gamma_0\}$. Since $\delta(n_G) = p^{n-|\Gamma_0|}$, the corresponding class number relation is

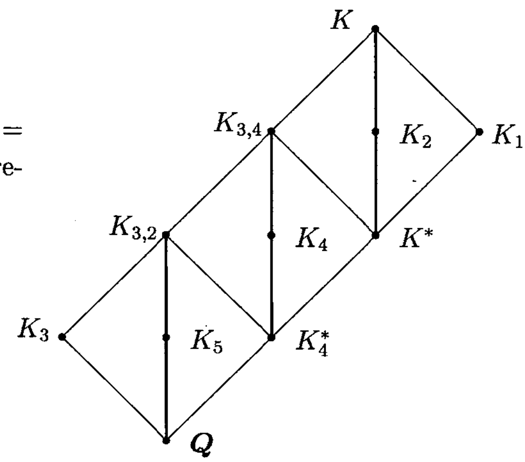
$$\frac{h_S(K)}{h_S(K^*)} = \frac{\prod_{H \in \Gamma_0} h_S(K^H)}{\left(\prod_{H^* \in \Gamma_0^*} h_S(K^{H^*})\right)^{p^{n-1}}} \cdot \frac{\delta(i_{E_S,M_S})}{\delta(L_{S_0})} \cdot p^{n-|\Gamma_0|}.$$

EXAMPLE 2. Set $p = 2$. Let q be a positive integer such that $8 \mid \phi(q)$. Let m be a square free integer prime to q. Let K_1 be the cyclic extension of degree 8 and of conductor q over $\boldsymbol{Q}$. Put $K = K_1(\sqrt{m})$ and $k = \boldsymbol{Q}$. We see $n = 2$, $m_1 = 8$ and $m_2 = 2$. Put $K_3 = \boldsymbol{Q}(\sqrt{m})$. We define $\chi_1 \sim \chi_2$ for $\chi_i \in \hat{G}$ to be $\langle \chi_1 \rangle = \langle \chi_2 \rangle$. Let ρ (*resp.* χ) be a Dirichlet character corresponding to K_1 (*resp.* K_3) of order 8 (*resp.* 2). A set of complete representatives of $Z/\sim$ is given by

$$\{\rho, \quad \rho\chi, \quad \rho^2\chi, \quad \rho^4\chi, \quad \chi\}.$$

For each character ζ, we associate a subgroup $H = \operatorname{Ker} \zeta$ and a subfield $L = K^H$ as follows.

ζ	ρ	$\rho\chi$	$\rho^2\chi$	$\rho^4\chi$	χ
H	H_1	H_2	H_4	H_5	H_3
L	K_1	K_2	K_4	K_5	K_3
$\lvert\zeta\rvert$	8	8	4	2	2



We observe $G^* = \langle \rho^2 \rangle^\perp$ and

$$H_1^* = H_2^* = G^*, H_3^* = H_5^* = G, H_4^* = \langle \rho^4 \rangle^\perp.$$

Write K_4^* for $K^{H_4^*}$. Theorem 10 yields a character relation

$$(22) \quad 0 = \psi_1 - \sum_{i=1}^{5} \psi_{H_i} + \psi_{G^*} + \psi_{H_4^*} + 2\psi_G.$$

Since $\delta(n_G) = 8^{-1}$, the corresponding class number relation is

$$(23) \quad \frac{h_S(K)h_S(K_4^*)h_S(K^*)}{\prod_{i=1}^5 h_S(K_i)} = \frac{\delta(i_{E_S,M_S})}{8\delta(L_{S_0})}.$$

We compute the norm relation γ^G and obtain

$$0 = s_G - \sum_{i=1}^{5} s_{H_i} + s_{H_{3,2}} + s_{H_{3,4}} + 2s_1,$$

where $H_{3,2}$ and $H_{3,4}$ are terms of the composition series

$$H_3 > H_{3,2} > H_{3,4} > \{1\}.$$

Denote by $K_{3,2}$ and $K_{3,4}$ the fixed field by $H_{3,2}$ and $H_{3,4}$, respectively. Since $\tilde{e}_H = |G|e_H$, the character relation

$$\begin{aligned} 0 &= \psi_1 - \psi_{H_1} - \psi_{H_2} - 4\psi_{H_3} \\ &\quad - 2\psi_{H_4} - 4\psi_{H_5} + 2\psi_{H_{3,2}} + \psi_{H_{3,4}} + 8\psi_G \end{aligned}$$

yields from Lemma 8. If we apply Theorem 10 to Galois extension $K_{3,4}/K_4^*$ and $K_{3,2}/\boldsymbol{Q}$ and lift the obtained character relations onto those of G, we can transform the above relation to (22).

We assume K_1 is real abelian and K_3 is imaginary quadratic and $S_0 = \emptyset$. We see K and K_2, K_3, K_4, K_5 are imaginary abelian field. By Corollary 7, we have

$$\frac{h^-(K)}{\prod_{i=2}^{5} h^-(K_i)} = \frac{\delta(Q^-)\delta(w^-)}{8}.$$

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